

ELEMENTARY MAPS ON TRIANGULAR ALGEBRAS

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ABSTRACT. In this note we prove that elementary surjective maps on triangular algebras are automatically additive.

The study of elementary maps was initiated by Brešar and Šemrl. Following ([1]), elementary maps are defined as follows.

Definition 1. Let \mathcal{R} and \mathcal{R}' be two rings. Suppose that $M: \mathcal{R} \rightarrow \mathcal{R}'$ and $M^*: \mathcal{R}' \rightarrow \mathcal{R}$ are two maps. Call the ordered pair (M, M^*) an *elementary map* of $\mathcal{R} \times \mathcal{R}'$ if

$$\begin{cases} M(aM^*(x)b) = M(a)xM(b), \\ M^*(xM(a)y) = M^*(x)aM^*(y) \end{cases}$$

for all $a, b \in \mathcal{R}$ and $x, y \in \mathcal{R}'$.

Notice that no additivity of M and M^* is required in the above definition.

It is very interesting that elementary maps on some rings as well as operator algebras are automatically additive. In recent years, the study of additivity of maps on rings and operator algebras has become an active topic which has enriched the theory of maps on rings and operator algebras.

The first result about the additivity of maps on rings was due to Martindale III. In [10], he proved the following result.

Theorem 2. ([10]) *Let \mathcal{R} be a ring containing a family $\{e_\alpha : \alpha \in \Lambda\}$ of idempotents which satisfies:*

- (i) $x\mathcal{R} = \{0\}$ implies $x = 0$;
- (ii) If $e_\alpha\mathcal{R}x = \{0\}$ for each $\alpha \in \Lambda$, then $x = 0$ (and hence $\mathcal{R}x = \{0\}$ implies $x = 0$);
- (iii) For each $\alpha \in \Lambda$, $e_\alpha x e_\alpha \mathcal{R}(1 - e_\alpha) = \{0\}$ implies $e_\alpha x e_\alpha = 0$.

Then any multiplicative bijective map from \mathcal{R} onto an arbitrary ring \mathcal{R}' is additive.

As a corollary, every multiplicative bijective map from a prime ring containing a nontrivial idempotent onto an arbitrary ring is necessarily additive.

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It should be mentioned here that the proof of [10] has become a quite standard argument and been applied widely in dealing with the additivity of a large number of maps on rings and operator algebras (see [3]-[9]).

The aim of this note is to study the additivity of elementary maps on triangular algebras. We will show that every elementary map on triangular operator is additive. Note that, different from [10], we do not require the existence of nontrivial idempotents. However, we still follow the line of [10].

Recall that a *triangular algebra* $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is an algebra of the form

$$Tri(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\}$$

under the usual matrix operations, where \mathcal{A} and \mathcal{B} are two algebras over a commutative ring \mathcal{R} , and \mathcal{M} is an $(\mathcal{A}, \mathcal{B})$ -bimodule which is faithful as a left \mathcal{A} -module and also as a right \mathcal{B} -module (see [2]).

We are ready to state our result of this note.

Theorem 3. *Let \mathcal{R}' be an arbitrary ring. Let \mathcal{A} and \mathcal{B} be two algebras over a commutative ring \mathcal{R} , \mathcal{M} a faithful $(\mathcal{A}, \mathcal{B})$ -bimodule, and \mathcal{T} be the triangular algebra $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$. Suppose that algebras \mathcal{A} and \mathcal{B} satisfy:*

- (i) *If $a\mathcal{A} = \{0\}$, or $\mathcal{A}a = \{0\}$, then $a = 0$;*
- (ii) *If $b\mathcal{B} = \{0\}$, or $\mathcal{B}b = \{0\}$, then $b = 0$.*

Suppose that (M, M^) is an elementary map on $\mathcal{T} \times \mathcal{R}'$, and both M and M^* are surjective. Then both M and M^* are additive.*

For the sake of clarity, we divide the proof into a series of lemmas. We begin with the following trivial one.

Lemma 4. $M(0) = 0$ and $M^*(0) = 0$.

Proof. We have $M(0) = M(0M^*(0)0) = M(0)0M(0) = 0$.

Similarly, $M^*(0) = M^*(0M(0)0) = M^*(0)0M^*(0) = 0$. □

In what follows, we set

$$\begin{aligned} \mathcal{T}_{11} &= \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathcal{A} \right\}, \\ \mathcal{T}_{12} &= \left\{ \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} : m \in \mathcal{M} \right\}, \end{aligned}$$

and

$$\mathcal{T}_{22} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} : b \in \mathcal{B} \right\}.$$

Then we may write $\mathcal{T} = \mathcal{T}_{11} \oplus \mathcal{T}_{12} \oplus \mathcal{T}_{22}$, and every element $a \in \mathcal{T}$ can be written as $a = a_{11} + a_{12} + a_{22}$. Note that notation a_{ij} denotes an arbitrary element of \mathcal{T}_{ij} .

The following result shows that both M and M^* are bijective.

Lemma 5. *Both M and M^* are injective.*

Proof. Suppose that $M(a) = M(b)$ for some a and b in \mathcal{T} . We write $a = a_{11} + a_{12} + a_{22}$ and $b = b_{11} + b_{12} + b_{22}$.

For arbitrary x and y in \mathcal{R}' , we have

$$M^*(x)aM^*(y) = M^*(xM(a)y) = M^*(aM(b)y) = M^*(x)bM^*(y).$$

This, by the surjectivity of M^* , is equivalent to

$$(1) \quad sat = sbt$$

for arbitrary $s, t \in \mathcal{T}$.

In particular, letting $s = s_{11}, t = t_{11} \in \mathcal{T}_{11}$ in equality (1), we get $s_{11}a_{11}t_{11} = s_{11}b_{11}t_{11}$. And so, by condition (i) in Theorem 3, $a_{11} = b_{11}$.

Similarly, we can get $a_{22} = b_{22}$ by letting $s = s_{22}$ and $t = t_{22}$ in identity (1).

We now show that $a_{12} = b_{12}$. Setting $s = s_{11} \in \mathcal{T}_{11}$ and $t = t_{22} \in \mathcal{T}_{22}$, then equality (1) becomes $s_{11}at_{22} = s_{11}bt_{22}$, i.e., $s_{11}a_{12}t_{22} = s_{11}b_{12}t_{22}$. Therefore $a_{12} = b_{12}$ as \mathcal{M} is a faithful $(\mathcal{A}, \mathcal{B})$ -bimodule.

To complete the proof, it remains to show that M^* is injective. Let x and y be in \mathcal{R}' such that $M^*(x) = M^*(y)$. Now for any $a, b \in \mathcal{T}$, we have

$$\begin{aligned} & M^*M(a)M^{-1}(x)M^*M(b) \\ &= M^*(M(a)MM^{-1}(x)M(b)) \\ &= M^*(M(a)xM(b)) \\ &= M^*M(aM^*(x)b) \\ &= M^*M(aM^*(y)b) \\ &= M^*(M(a)yM(b)) \\ &= M^*(M(a)MM^{-1}(y)M(b)) \\ &= M^*M(a)M^{-1}(y)M^*(b). \end{aligned}$$

Thus

$$M^*M(a)M^{-1}(x)M^*M(b) = M^*M(a)M^{-1}(y)M^*M(b).$$

Equivalently,

$$sM^{-1}(x)t = sM^{-1}(y)t$$

for any $s, t \in \mathcal{T}$ since M^*M is surjective.

It follows from the same argument above that $M^{-1}(x) = M^{-1}(y)$, and so $x = y$, as desired. \square

Lemma 6. *The pair (M^{*-1}, M^{-1}) is a Jordan elementary map on $\mathcal{T} \times \mathcal{R}'$. That is,*

$$\begin{cases} M^{*-1}(aM^{-1}(x)b) = M^{*-1}(a)xM^{*-1}(b), \\ M^{-1}(xM^{*-1}(a)y) = M^{-1}(x)aM^{-1}(y) \end{cases}$$

for all $a, b \in \mathcal{T}$ and $x, y \in \mathcal{R}'$.

Proof. The first identity follows from the following observation.

$$\begin{aligned} & M^*(M^{*-1}(a)xM^{*-1}(b)) \\ &= M^*(M^{*-1}(a)MM^{-1}(x)M^{*-1}(b)) \\ &= aM^{-1}(x)b. \end{aligned}$$

The second one goes similarly. \square

The following result will be used frequently in this note.

Lemma 7. *Let $a, b, c \in \mathcal{R}$ such that $M(c) = M(a) + M(b)$. Then for any $s, t \in \mathcal{T}$ we have*

$$M^{*-1}(sct) = M^{*-1}(sat) + M^{*-1}(sbt).$$

Proof. By Lemma 6, we have

$$\begin{aligned} M^{*-1}(sct) &= M^{*-1}(sM^{-1}M(c)t) \\ &= M^{*-1}(s)M(c)M^{*-1}(t) \\ &= M^{*-1}(s)(M(a) + M(b))M^{*-1}(t) \\ &= (M^{*-1}(s)M(a)M^{*-1}(t)) + (M^{*-1}(s)M(b)M^{*-1}(t)) \\ &= M^{*-1}(sat) + M^{*-1}(sbt). \end{aligned}$$

\square

Lemma 8. *Let $a_{11} \in \mathcal{T}_{11}$ and $b_{12} \in \mathcal{T}_{12}$, then*

- (i) $M(a_{11} + b_{12}) = M(a_{11}) + M(b_{12})$;
- (ii) $M^{*-1}(a_{11} + b_{12}) = M^{*-1}(a_{11}) + M^{*-1}(b_{12})$.

Proof. We only prove (i). We choose $c \in \mathcal{T}$ such that $M(c) = M(a_{11}) + M(b_{12})$. For arbitrary $s_{11} \in \mathcal{R}_{11}$ and $t_{22} \in \mathcal{T}_{22}$, by Lemma 7, we have

$$M^{*-1}(s_{11}ct_{22}) = M^{*-1}(s_{11}a_{11}t_{22}) + M^{*-1}(s_{11}b_{12}t_{22}) = M^{*-1}(s_{11}b_{12}t_{22}).$$

It follows that $s_{11}ct_{22} = s_{11}b_{12}t_{22}$, i.e., $s_{11}c_{12}t_{22} = s_{11}b_{12}t_{22}$, which yields that $c_{12} = b_{12}$.

Now for any s_{11} and t_{11} in \mathcal{T}_{11} , we have

$$M^{*-1}(s_{11}ct_{11}) = M^{*-1}(s_{11}a_{11}t_{11}) + M^{*-1}(s_{11}b_{12}t_{11}) = M^{*-1}(s_{11}a_{11}t_{11}).$$

This implies that $c_{11} = a_{11}$.

Similarly, for any $s_{22}, t_{22} \in \mathcal{T}_{22}$, we obtain

$$M^{*-1}(s_{22}ct_{22}) = M^{*-1}(s_{22}a_{11}t_{22}) + M^{*-1}(s_{22}b_{12}t_{22}) = 0.$$

Hence $c_{22} = 0$ follows from the fact that $s_{22}ct_{22} = 0$. \square

Similarly, we can get the following result.

Lemma 9. *Let $a_{22} \in \mathcal{T}_{22}$ and $b_{12} \in \mathcal{T}_{12}$, then*

- (i) $M(a_{22} + b_{12}) = M(a_{22}) + M(b_{12})$;
- (ii) $M^{*-1}(a_{22} + b_{12}) = M^{*-1}(a_{22}) + M^{*-1}(b_{12})$.

Lemma 10. *For any $t_{11}, a_{11} \in \mathcal{T}_{11}$, $b_{12}, c_{12} \in \mathcal{T}_{12}$, and $d_{22} \in \mathcal{T}_{22}$, we have*

- (i) $M(t_{11}a_{11}b_{12} + t_{11}c_{12}d_{22}) = M(t_{11}a_{11}b_{12}) + M(t_{11}c_{12}d_{22})$;
- (ii) $M^{*-1}(t_{11}a_{11}b_{12} + t_{11}c_{12}d_{22}) = M^{*-1}(t_{11}a_{11}b_{12}) + M^{*-1}(t_{11}c_{12}d_{22})$.

Proof. We only prove (i). Using Lemma 8 and Lemma 9, we compute

$$\begin{aligned}
& M(t_{11}a_{11}b_{12} + t_{11}c_{12}d_{22}) \\
&= M(t_{11}(a_{11} + c_{12})(b_{12} + d_{22})) \\
&= M(t_{11}M^*M^{*-1}(a_{11} + c_{12})(b_{12} + d_{22})) \\
&= M(t_{11})M^{*-1}(a_{11} + c_{12})M(b_{12} + d_{22}) \\
&= M(t_{11})M^{*-1}(a_{11})M(b_{12}) + M(t_{11})M^{*-1}(a_{11})M(d_{22}) \\
&\quad + M(t_{11})M^{*-1}(c_{12})M(b_{12}) + M(t_{11})M^{*-1}(c_{12})M(d_{22}) \\
&= M(t_{11})(M^{*-1}(a_{11}) + M^{*-1}(c_{12}))M(b_{12}) + M(t_{11})(M^{*-1}(a_{11}) + M^{*-1}(c_{12}))M(d_{22}) \\
&= M(t_{11})M^{*-1}(a_{11} + c_{12})M(b_{12}) + M(t_{11})M^{*-1}(a_{11} + c_{12})M(d_{22}) \\
&= M(t_{11}(a_{11} + c_{12})b_{12}) + M(t_{11}(a_{11} + c_{12})d_{22}) \\
&= M(t_{11}a_{11}b_{12}) + M(t_{11}c_{12}d_{22}).
\end{aligned}$$

□

Lemma 11. *Both M and M^{*-1} are additive on \mathcal{T}_{12} .*

Proof. Let a_{12} and b_{12} be in \mathcal{T}_{12} . We pick $c \in \mathcal{T}$ such that $M(c) = M(a_{12}) + M(b_{12})$.

For arbitrary $t_{11}, s_{11} \in \mathcal{T}_{11}$, by Lemma 7, we have

$$M(t_{11}cs_{11}) = M(t_{11}a_{12}s_{11}) + M(t_{11}b_{12}s_{11}) = 0,$$

this implies that $t_{11}cs_{11} = 0$, and so $c_{11} = 0$.

Similarly, we can get $c_{22} = 0$.

We now show that $c_{12} = a_{12} + b_{12}$. For any $t_{11}, r_{11} \in \mathcal{T}_{11}$ and $s_{22} \in \mathcal{T}_{22}$, by Lemma 7 and Lemma 10, we obtain

$$\begin{aligned}
M(r_{11}t_{11}cs_{22}) &= M(r_{11}t_{11}a_{12}s_{22}) + M(r_{11}t_{11}b_{12}s_{22}) \\
&= M(r_{11}t_{11}a_{12}s_{22} + r_{11}t_{11}b_{12}s_{22}) \\
&= M(r_{11}t_{11}(a_{12} + b_{12})s_{22}).
\end{aligned}$$

It follows that

$$r_{11}t_{11}cs_{22} = r_{11}t_{11}(a_{12} + b_{12})s_{22}.$$

Equivalently,

$$r_{11}t_{11}c_{12}s_{22} = r_{11}t_{11}(a_{12} + b_{12})s_{22}.$$

Then we get $c_{12} = a_{12} + b_{12}$.

With the similar argument, one can see that M^{*-1} is also additive on \mathcal{T}_{12} . \square

Lemma 12. *Both M and M^{*-1} are additive on \mathcal{T}_{11} .*

Proof. We only show the additivity of M on \mathcal{T}_{11} . Suppose that a_{11} and b_{11} are two elements of \mathcal{T}_{11} . Let $c \in \mathcal{T}$ be chosen satisfying $M(c) = M(a_{11}) + M(b_{11})$. Now for any $t_{22}, s_{22} \in \mathcal{T}_{22}$, we have

$$M(t_{22}cs_{22}) = M(t_{22}a_{11}s_{22}) + M(t_{22}b_{11}s_{22}) = 0.$$

Consequently, $t_{22}cs_{22} = 0$, i.e., $t_{22}c_{22}s_{22} = 0$, and so $c_{22} = 0$.

Similarly, we can infer that $c_{12} = 0$.

To complete the proof, we need to show that $c_{11} = a_{11} + b_{11}$. For each $t_{11} \in \mathcal{T}_{11}$ and $s_{12} \in \mathcal{T}_{12}$, we consider

$$M(t_{11}cs_{12}) = M(t_{11}a_{11}s_{12}) + M(t_{11}b_{11}s_{12}) = M(t_{11}a_{11}s_{12} + t_{11}b_{11}s_{12}).$$

Note that in the last equality we apply Lemma 11. It follows that

$$t_{11}cs_{12} = t_{11}a_{11}s_{12} + t_{11}b_{11}s_{12}.$$

This leads to $c_{11} = a_{11} + b_{11}$, as desired. \square

Lemma 13. *M and M^{*-1} are additive on \mathcal{T}_{22} .*

Proof. Suppose that a_{22} and b_{22} are in \mathcal{T}_{22} . We choose $c \in \mathcal{T}$ such that $M(c) = M(a_{22}) + M(b_{22})$. For any $t_{12} \in \mathcal{T}_{12}$ and $s_{22} \in \mathcal{T}_{22}$, using Lemma 11, we have

$$M(t_{12}cs_{22}) = M(t_{12}a_{22}s_{22}) + M(t_{12}b_{22}s_{22}) = M(t_{12}(a_{22} + b_{22})s_{22}).$$

Accordingly, $t_{12}cs_{22} = t_{12}(a_{22} + b_{22})s_{22}$, which yields that $c_{22} = a_{22} + b_{22}$.

With the similar argument, we can verify that $c_{11} = c_{12} = 0$.

The additivity of M^{*-1} on \mathcal{T}_{22} follows easily. \square

Lemma 14. *For any $a_{11} \in \mathcal{T}_{11}$, $b_{12} \in \mathcal{T}_{12}$, and $c_{22} \in \mathcal{T}_{22}$, the following are true.*

- (i) $M(a_{11} + b_{12} + c_{22}) = M(a_{11}) + M(b_{12}) + M(c_{22})$;
- (ii) $M^{*-1}(a_{11} + b_{12} + c_{22}) = M^{*-1}(a_{11}) + M^{*-1}(b_{12}) + M^{*-1}(c_{22})$.

Proof. We only prove (i). Let $d \in \mathcal{T}$ be an element satisfying $M(d) = M(a_{11}) + M(b_{12}) + M(c_{22})$. For any $s, t \in \mathcal{T}$, using Lemma 7 twice, we can arrive at

$$(2) \quad M(sdt) = M(sa_{11}t) + M(sb_{12}t) + M(sc_{22}t).$$

Letting $s = s_{11}$ and $t = t_{11}$ in the above equality, we get $d_{11} = a_{11}$.

In the same fashion for $s = s_{22}$ and $t = t_{22}$ in equality (2), we can infer that $d_{22} = c_{22}$.

Finally, considering $s = s_{11}$ and $t = t_{22}$ in equality (2), we see that $d_{12} = b_{12}$. Thus, $d = a_{11} + b_{12} + c_{22}$, which completes the proof. \square

We now prove our main result.

Proof of Theorem 3 We first show the additivity of M . Let $a = a_{11} + a_{12} + a_{22}$ and $b = b_{11} + b_{12} + b_{22}$ be two arbitrary elements of \mathcal{T} . We have

$$\begin{aligned}
 M(a + b) &= M((a_{11} + b_{11}) + (a_{12} + b_{12}) + (a_{22} + b_{22})) \\
 &= M(a_{11} + b_{11}) + M(a_{12} + b_{12}) + M(a_{22} + b_{22}) \\
 &= M(a_{11}) + M(b_{11}) + M(a_{12}) + M(b_{12}) + M(a_{22}) + M(b_{22}) \\
 &= (M(a_{11}) + M(a_{12}) + M(a_{22})) + (M(b_{11}) + M(b_{12}) + M(b_{22})) \\
 &= M(a_{11} + a_{12} + a_{22}) + M(b_{11} + b_{12} + b_{22}) \\
 &= M(a) + M(b).
 \end{aligned}$$

That is, M is additive.

We now turn to prove that M^* is additive. For any $x, y \in \mathcal{R}'$, there exist $c = c_{11} + c_{12} + c_{22}$ and $d = d_{11} + d_{12} + d_{22}$ in \mathcal{R} such that $c = M^*(x + y)$ and $d = M^*(x) + M^*(y)$.

For arbitrary $s, t \in \mathcal{T}$, by the additivity of M , we compute

$$\begin{aligned}
 M(sct) &= M(sM^*(x + y)t) \\
 &= M(s)(x + y)M(t) \\
 &= M(s)xM(t) + M(s)yM(t) \\
 &= M(sM^*(x)t) + M(sM^*(y)t) \\
 &= M(sM^*(x)t + sM^*(y)t) \\
 &= M(s(M^*(x) + M^*(y))t) \\
 &= M(sdt),
 \end{aligned}$$

which implies that $sct = sdt$. Furthermore, we get $c = d$, i.e., $M^*(x + y) = M^*(x) + M^*(y)$.

In particular, if both \mathcal{A} and \mathcal{B} are unital algebras, we have

Corollary 15. *Let \mathcal{R}' be an arbitrary ring. Let \mathcal{A} and \mathcal{B} be two unital algebras over a commutative ring \mathcal{R} , \mathcal{M} a faithful $(\mathcal{A}, \mathcal{B})$ -bimodule, and \mathcal{T} be the triangular algebra $\text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$. Suppose that (M, M^*) is an elementary map on $\mathcal{T} \times \mathcal{R}'$, and both M and M^* are surjective. Then both M and M^* are additive.*

Recall that a subalgebra of $B(X)$ is called a *standard operator* if it contains all finite rank operators, where $B(X)$ is the algebra of all bounded linear operator on a Banach space X .

We complete this note by considering elementary maps on triangular algebras provided \mathcal{A} and \mathcal{B} are standard operator algebras.

Corollary 16. *Let \mathcal{R}' be an arbitrary ring. Let \mathcal{A} and \mathcal{B} be two standard operator algebras over a Banach space X , \mathcal{M} a faithful $(\mathcal{A}, \mathcal{B})$ -bimodule, and \mathcal{T} be the triangular algebra $\text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$. Suppose that (M, M^*) is an elementary map on $\mathcal{T} \times \mathcal{R}'$, and both M and M^* are surjective. Then both M and M^* are additive.*

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